ELASTICITY

1.

With the behavior law we now have all the tools required to carry out a complete mechanical study. We will do these studies within the very common, and very simple, framework of linear elasticity. We will use this behavior law assuming that the medium is isotropic, which will make things more simple yet.

2.

The relations are valid if the material is homogeneous, isotropic, if the transformation is continuous, infinitesimal, mono thermal reversible, if the domain is subjected to no chemical transformation or state change and if the behavior is linear:

We clearly see the contributions from the initial stress state and temperature. However we will often consider that the initial stresses are negligible and the temperature is constant. This gives even more simple formulae

It is also possible to give equivalent index formulae. The medium being isotropic, these relations are valid in any basis. It is to be noted that it is possible to calculate the stress state from the knowledge of the deformation state and reversely.

It is obvious that for this behavior law, only two coefficients are independent; to the extent that we have relations between the two Lamé coefficients, Young’s modulus and Poisson’s ratio. It is to be noted that the first Lamé coefficient is often called Coulomb’s modulus or transversal elasticity modulus in opposition to Young’s modulus which is also called the longitudinal elasticity modulus.

3.

These various coefficients mainly depend on the material used. However, they may also depend on parameters such as temperature, for instance. Lamé coefficients and Young’s modulus are homogeneous with pressures and are generally expressed in Mega Pascal. Poisson’s ratio is dimensionless.

4.

We now have all the equations required to find the unknowns but sometimes, to ensure optimum efficiency, we can use complementary equations obtained by combining basis equations. This applies to Navier equations which express the fundamental principle of mechanics as a function of the displacement field. In order to obtain them, we will start from the fundamental principle of mechanics classically expressed in stresses, and then use the behavior law to get an expression according to the deformations. Finally using the displacement/deformation relations leads to the result expected.

Therefore we start from the resultant equation drawn from the fundamental principle of mechanics:

Introducing the linear elastic behavior law for an isotropic material makes it possible to obtain a new formulation:

But we also have fundamental relations drawn from the tensorial analysis:

This makes it possible to obtain the final form called Navier equations. Using the tensorial analysis, we can give two equivalent expressions of it.

We can also make a demonstration using the index notation. By working in a Cartesian coordinate system, it is possible to link the stress tensor component differential to the deformation tensor component differentials.

This gives an index expression representing the fundamental principle of mechanics expressed in deformations.

It is now sufficient to use the relations between the deformation tensor and the displacement vector to get in just a few steps the index relation expressing Navier equations.

This index formulation is equivalent to the tensorial formulation.

5.

A scalar relation can also be inferred by applying the divergence operator:

It is then possible to use relations resulting from the tensorial analysis:

And we ultimately get a scalar equation which does not completely replace Navier equation of course but will be used sometimes however.

From this equation, we can see that in the c ase of a constant volumetric mass domain, in equilibrium and placed in a zero divergence force field, the Laplacian of the displacement vector divergence is zero.

6.

Another set of additional equations is given by Beltrami equations which will express the deformation compatibility conditions as functions of the stress tensor components.

The latter have a relatively complex form when they are given as functions of the deformation tensor.

In order to introduce the stress tensor, we use the behavior law in the form of Hooke’s equations.

From this formulation it is possible to establish a link between the deformation and stress tensor traces.

We then use the scalar form of NAVIER equations

We also have tensorial analysis formulae

And to complete the picture, we can use the fundamental principle of mechanics

This makes it possible to get a new formulation of the compatibility equations

Equations which may also write in an index form

As with Navier equations, we get a simplified formulation in the case of a constant volumetric mass domain, in equilibrium and placed in a zero divergence volumetric force field.

7.

The purpose of the mechanical studies we have to do is to dimension the structures according to the changes applied. But in order to save energy, it is necessary to avoid using too much material, which leads to minimum dimensioning. To ensure this, we have to approach the behavior limits of our material without going beyond its boundaries. This leads to express these boundaries which will be determined by experimentation.

We could define these boundaries in the deformation state, but we actually express them according to the stress state. As we have seen that the latter was mostly determined in a main basis by means of its eigenvalues, it can be envisaged that the expression determining the boundary is a relation between the main stresses.

This amounts to saying that it is possible to work in a three dimensional space, the coordinates on the basis axes being the eigenvalues of the stress state studied. Therefore to each stress state a point in this space can be linked.

In particular, the system origin point is linked to a zero stress state.

And in that space there is a closed surface including necessarily the origin point and defining the permissive domain of the stress state. Any point located in the domain bounded by the closed surface represents an authorized stress state. But when this point is located on the envelope surface, we can consider that we are in the limit state.

The problem is that this representation is three dimensional and therefore difficult to use. That is the reason why we will try to determine this envelope surface in a two-dimensional representation, that of Mohr’s plane in which any stress state will be represented by a tri-circle.

8.

In order to determine this envelope surface, we will use tests and experiments. The simplest result to highlight is the one obtained by means of a single-axis tensile test. This test makes it possible to obtain at certain points of the specimen a very simple stress tensor, characterized by a non-zero value, that of the normal stress in the tensile direction.

It is easy to get the main three stresses, two of which are zero.

This gives a very specific Mohr’s tri-circle since two circles are merged and the third one is a zero-radius circle located on the system origin point.

Experimentally we can see that when a certain stress, said to be tensile elasticity limit, has been reached, our material enters a plastic domain characterized by irreversible deformations. Therefore there is a boundary Mohr’s tri-circle characterized by this tensile elasticity limit.

Therefore, as long as our tensile stress is under that tensile elasticity limit, the domain remains with the linear elastic behavior law. As a consequence, any single-axis tensile state giving a tri-circle inside the previous circle is considered to have a linear elastic behavior law.

9.

Symmetrically, we can address the single-axis compression test of a specimen. At certain points of the specimen, the stress tensor is of the same form as with single-axis tensile strength, but this time the normal stress in the load direction is negative.

Mohr’s tri-circle is then located in the negative part of the normal stresses.

And we also get a boundary tri-circle. Generally speaking, the single-axis compression elasticity limit is, in absolute terms, greater than the single-axis tensile elasticity limit.

10.

A shear load can be obtained for instance with a specimen in the form of a thin wall circular tube subjected to torsion. In a natural cylindrical system, the stress tensor shows one tangent component only.

It is easy then to specify the components of this stress tensor in the Eigen vector basis.

Mohr’s tri-circle form can be inferred from this.

In practice, it is noted again that, if we want to stay in the linear elasticity domain, the load prescribed to the specimen cannot exceed a certain limit value, which prescribes a maximum radius to the largest circle.

11.

The isotropic compression load is obtained by applying a uniform pressure on the outer surface bounding the study specimen. We then get at any point of the domain a stress tensor proportional to the identity tensor.

Mohr’s tri-circle is simply a point located on Mohr’s plane normal axis in the negative part.

In concrete terms, it is impossible to show any limitation. Whatever the pressure applied, when it is suppressed, the domain returns to its initial shape and dimensions. We are therefore constantly in the linear elastic field.

12.

We can see that, depending on the load, there are several answers. From all these results is it possible to standardize the answer? This is what Mohr Caquot’s elastic limit criterion attempts to do. In order to express it, it is sufficient to start from the set of boundary circles previously obtained.

The idea is that, in Mohr’s plane, there is an envelope curve of all these boundary conditions as well as others obtained with combined tension and torsion tests for instance.

With a load generating a Mohr’s tri-circle inside the envelope curve, the domain remains in the linear elastic field.

On the opposite, as soon as the tri-circle intersects the envelope curve, this means that we have left the linear elastic behavior law. It is then necessary either to recalculate with a new behavior law, or modify the load or structure to stay inside the boundary.

This elastic limit criterion solution is certainly a good answer but it is necessary to establish the boundary curve, which requires many tests and this is too costly for industrial effective use. It is preferable to find other solutions to establish the linear elasticity behavior law limits of use. .

13.

Among these other solutions, Von Misès criterion is one of the most commonly used. The idea is that the material can store deformation energy up to a certain limit only.

We are familiar with the differential expression of this deformation energy.

Integration is simple with a linear elastic behavior law expressing proportionality between the stress state and deformation state

However a simple limitation of the deformation energy does not permit to take into account the results of the isotropic compression test for which there is no loading limit. As this test is expressed by a purely spherical stress state, without a deviatory part, it is suggested to decompose the deformation energy into a spherical part and a deviatory part. It is to be noted that there is a complete separation of these two contributions.

To define the elasticity limit, Von Misès criterion is expressed by an inequality.

Using the tensile test results, we can set the value of the constant limiting the deviatory deformation energy:

This gives a relatively simple expression according to the main stresses.

Another criterion can be obtained by considering that shearing mainly generates plastic deformations. Thus it seems to be natural to limit the value of the maximum tangent stress actually corresponding to the radius of the largest Mohr’s circle. We then get Tresca’s criterion. These criteria are very simple to establish and implement. We get numerical expressions which can easily be treated by means of calculation software. But of course simplifying means loss in efficiency and there are instances in which these criteria are defective. But generally speaking, these cases correspond to states with very high isotropic tensile components, loads very difficult to apply and very rare to generate in real life. In the fields with very strong compression components, these criteria are generally reassuring.

14.

A priori we now have all the tools required to address a mechanics problem. We have 15 scalar functions to determine: the three components of the displacement vector, the six components of the deformation tensor and the six components of the stress tensor. With the behavior of a solid, the mass is not really an unknown.

We have 15 equations at our disposal. There are six scalar relations between the displacement vector and the deformation tensor. The resultant equation drawn from the fundamental principle of mechanics gives 3 scalar consequences. Finally we have completed this with the six equations derived from our behavior law. The continuity equation expressing mass conservation is mostly useful for fluid or gaseous elements, but is of little interest with the behavior of a solid. However, even if we have the right number of equations with respect to the number of unknowns, there is a huge difficulty left. These equations are not linear and unfortunately, there is no way of solving them analytically. We can for instance use a finite element numerical method to give sufficiently accurate answers. But it is necessary to adjust this numerical process and to do so, it is good to know analytical answers for a few specific cases at least.

15.

Determining these solutions requires simplifications, but it is also based upon the fact that the answer exists and is unique. Such is the case when a statistically acceptable stress field is linked to a kinematically acceptable deformation field. Let us examine what these new terms mean.

Consider the set of stress fields.

Within this set, some do not respect the fundamental principle of mechanics, but we will address specifically those in agreement with this principle, of course.

On the other hand, we also have a subset designating the stress field in agreement with the loading conditions prescribed to the surface of the domain studied.

And of course the stress field sought has to satisfy these two conditions. All such stress fields are said to be statistically acceptable.

16.

In a dual way, we can address the set of deformation fields.

There is a subset representing the deformation fields in agreement with the compatibility conditions.

And there is also a subset in which we find the deformation fields permitting to satisfy the boundary conditions on the displacements prescribed to the surface of the domain studied.

Once again only the deformation fields respecting these two conditions are the ones we are interested in. They are referred to as kinematically acceptable deformation fields.

Using the behavior law, we can define new couples between these two large sets. With a problem for which at each point of the surface bounding the domain we can define either the loading, or the displacement, we demonstrate that there is only one couple made of a statistically acceptable stress tensor and a kinematically acceptable deformation tensor, these two tensorial states being interconnected by the behavior law. This is the solution unicity theorem. Proof by the absurd can make the demonstration of this theorem. Indeed if we admit that there are two different solutions for a given load, by applying the superposition theorem and making the difference between these two solutions, we get a non zero stress state for a zero load.

17.

With the unicity theorem, we can envisage a method to determine the solution of a mechanics problem based upon the notion of resolution schemes. The idea is to start from assumptions specific to the study permitting to reduce the number of unknowns and parameters. For instance in a perfectly axisymmetric problem, we can consider that the angular position angle is not a parameter of this problem. Then we try and demonstrate that these assumptions make it possible to have a statistically acceptable stress field connected to a kinetically acceptable deformation field by means of the behavior law. This involves a few calculations and verifications. If we can get such a binomial, then, by means of the unicity theorem, we can say that we have the solution to the problem. There are several possible resolution schemes.

It is possible for instance to formulate assumptions on the stress state.

We then verify the fundamental principle of mechanics.

If the assumptions made do not permit to respect this principle, they have to be re-examined.

On the contrary, if there is no incompatibility, it is possible to continue by testing whether the boundary conditions on the forces prescribed to the surface of the domain are respected.

If the test is positive, the stress tensor being statically acceptable, it is then possible to calculate the components of the deformation tensor using Hooke's law.

We then examine whether the tensor obtained respects the compatibility equations.

Finally we test the boundary conditions in prescribed displacements.

If both last tests are positive, since the deformation tensor is kinematically acceptable, we then have a solution to our problem. It is to be noted that this resolution scheme can be simplified if there is no boundary condition on the displacements.

In this case indeed, it is sufficient to validate the compatibility equations by verifying them in the form of stresses, i.e. using Beltrami equations.

It is also possible to have a resolution scheme by making assumptions on the displacement field.

Then the boundary conditions on the displacements are verified.

It is then possible to calculate the deformation tensor components. As this tensor is obtained from a displacement field, it is not necessary to verify the compatibility equations. At this point, the deformation state is kinematically acceptable.

It is then possible to define the stress tensor using the Lamé laws.

It is then possible to test the fundamental principle of mechanics.

And conclude with the verification of the boundary conditions on the forces.

If all the tests are positive, we have the solution to the problem. If, from the assumptions made, it is not possible to verify one of the tests of the scheme used, then these assumptions have to be re-examined.

18.

Let us now examine the way in which this method makes it possible to find the stress and deformation state in the case of a homogeneous body with no specific shape subjected to a constant pressure applied to its outer surface. By prescribing this pressure, we note that the body does not change shape, but volume only. This will permit to build our resolution scheme.

The starting assumption is about the deformation state.

The fact that there is no shape change in the load application leads to imagine that the deformation tensor deviatory part is zero.

Therefore our deformation tensor is purely spherical, i.e. proportional to the identity tensor. A priori the proportionality coefficient depends on the point studied.

As there is little we can do, let us examine the effects on the stress tensor. The latter is obtained by applying the Lamé law.

The trace of the deformation tensor is very simple to get and we finally note that the stress tensor is also proportional to the identity tensor.

We can now examine the effects of the fundamental principle of mechanics.

The domain is in equilibrium; therefore the term connected to acceleration is zero.

Besides, as the only load is the one prescribed by the pressure on the domain surface, we can consider that the remote volumetric actions are zero.

The fundamental principle of mechanics shows that the divergence of the stress tensor is zero. This induces three scalar consequences showing that the stress tensor is the same at any point of the domain.

We can address the force boundary condition which is expressed by the fact that, at any point of the domain outer surface, the stress vector according to the normal exterior to the domain is equal to the pressure vector applied to this point.

This enables us to determine the constant value

With these two conditions verified, the stress tensor is statically acceptable. We must ensure that the deformation tensor is kinetically acceptable. To do so, it has to verify the compatibility equations.

Although these equations are in a complex form, the verification is immediate and straightforward for the deformation tensor is constant and the terms involved in these equations are second derivatives of the components of this tensor.

We should now examine the boundary conditions on the displacements prescribed to the domain surface, but there are not any so they are not crucial to the study. Therefore we can say that we have the solution to our problem.

We can see that this solution gives a very simple expression of the stress tensor.

19.

In a second application, we can try and find the solution of an envelope under pressure. It is a straight circular cross-section tube, of limited length. From the dimensional viewpoint, it is characterized by an inner radius Ri, an outer radius Re and a height H.

For any load, a pressure Pi is applied to the cylindrical inner wall whereas a pressure Pe is applied to the cylindrical outer wall.

20.

As a result of this load, there will be deformations and these will be mainly generated by the radial displacements occurring at any point of the material. Thus we will place ourselves in a natural cylindrical system in order to assume that the orthoradial and radial components of the displacement field are zero.

Besides, for symmetry reasons, we consider that the angular and axial position variables are not state variables. As a consequence, the displacement radial component is a function of the radius of the point considered only. With these assumptions, instead of having a displacement field characterized by three scalar functions each of which depends on a single variable, we have a single scalar function depending on a single variable. We now have to verify if, with these assumptions, we get a kinematically acceptable deformation field connected to a statically acceptable stress field by means of the behavior law.

To do so, we will first calculate the components of the deformation tensor. The latter represents the symmetric part of the displacement field gradient tensor. In order to use this relation correctly, we can express the displacement field components in a Cartesian coordinate system and use the index relations seen earlier. But it is also possible to use the set of the displacement deformation relations in cylindrical coordinates. It is to be noted that, contrary to what one may think in the first place, the linear dilation in the circumference direction is non zero. On second thoughts however, we realize that this is normal since the circumference varies according to the pressure values.

It is to be noted that, since we have no boundary condition prescribing displacements, the deformation tensor is kinematically acceptable. To calculate the stress tensor, we use the Lamé formulae.

This gives the stress tensor components. Since we are in the main directions, we get a diagonal tensor. Of course, everything depends on the displacement field radial component. We can also note that the axial normal stress is non zero.

21.

To validate this solution, we have to verify that the stress tensor we have just obtained is statically acceptable. To do so, let us first examine whether the fundamental principle of mechanics is satisfied in the local form. As the domain is in equilibrium, the acceleration vector is zero. The same applies to the vector representing remote actions for we have no volumetric force prescribed. The local equation of the resultant from the fundamental principle of mechanics then takes a simple form.

Of course, in order to express this equation correctly, it is good to take the expressions in cylindrical coordinates, which brings to our projections on to the axis system non-trivial components. It is to be noted that only the first of these three projections brings about interesting elements, the other two being automatically satisfied.

We then express this first equation according to the displacement field components. This leads to a second order differential equation with respect to the displacement field radial component.

Solving this differential equation leads to the displacement forms in agreement with the fundamental principle of mechanics. They are defined by two integration constants.

It is then possible to calculate the stress tensor components.

In order to ensure that the stress tensor is statically acceptable, it is sufficient to verify the boundary conditions of the stresses prescribed to the domain surfaces. It is necessary to express the pressures applied to the inner and outer cylindrical surfaces, which gives two vectorial conditions.

22.

Using these conditions, we can then calculate the integration constants, especially those connected to the stress tensor components.

This makes it possible to get the two integration constants defining the displacement field:

Once these constants have been calculated, we might think that the process is over and that we can end on this result. But we must remember that it is important to express all the boundary conditions. In this case, we have to use the fact that all the end surfaces of our cylinder are not loaded, which gives additional vectorial conditions

These give a single relation related to the constants. But this result is incompatible with the previous ones obviously. This means that our starting assumptions are wrong. On second thoughts, it appears that the displacement field must have an axial component mainly due to the effects of Poisson’s ratio. For readers who do not want to stay in the dark with respect to this application, they can find the answers to this problem in the course handout.