BEHAVIOR

1.

It is easy to understand that for a given geometry and a change prescribed, a structure will deform more or less according to the material used. We will now study this property called behavior.

2.

In the study of structures, it is readily apparent that applying loads generates deformations and displacements. Sometimes the latter are slight, invisible to the naked eye, but always measurable. This is called direct control.

But it is also possible to prescribe displacements to our structure. Obviously, in order to get a certain deformed state, it will be necessary to apply relevant loads. This is called displacement control.

And reality may be slightly more complex. Indeed the force applied to the structure may be independent from the displacement.

3.

We will show that it is necessary to connect the kinematic and static aspects by means of a mathematical balance of the unknowns to determine and the equations at our disposal.

With regard to the problem unknown, we can start with the mass of the domain studied, which gives a scalar function to determine.

We then encountered the displacement field, i.e. the displacement vector at any point of our domain. As it is a vectorial field, it is determined by three scalar functions.

This allowed us to introduce the notion of deformation tensor which, as it is symmetric, is characterized by six scalar functions.

After this kinematic study, we moved to the dynamic approach, which led us to the stress tensor which is symmetric as well and therefore determined by six scalar functions.

Therefore we have sixteen functions to determine altogether at any point of our domain.

4.

We will now examine the equations at our disposal.

To express mass conservation, we have the continuity equation.

In the kinematic study, we have established the connections between the displacement vector and the deformation tensor.

Finally in the dynamic study, the fundamental principle of mechanics gives three scalar relations by means of the resultant equation. The moment equation is already taken into account by specifying that the stress tensor is symmetric.

Therefore we can see that, as things stand, we do not have enough equations to find all our unknowns. However there is a solution, so we just have to make up for that. We will do so by means of the equations resulting from the material behavior law.

5.

These relations cannot be independent from the laws of physics, and especially thermodynamics. As the latter permits, among other things, to establish connections between various forms of energy, we will give a local expression of the kinetic energy theorem.

The demonstration starts from the local expression of the fundamental principle of mechanics.

The scalar product with the velocity vector gives homogeneous terms to a power.

The first term also appears in a divergence formula.

With the symmetry of the stress tensor, we can show the double product of the stress tensor by the deformation rate tensor.

We thus get a local expression expressing the kinetic energy theorem.

This can then be globalized by integration over the whole study domain.

6.

Using the particle differential of an integral, the continuity equation and the divergence theorem permits to get a new expression. Let us analyze the various terms of the latter formulation.

In the first member of the equality we can find the differential with respect to time of the Galilean kinetic energy of our domain.

The first term of the second member represents the power of the outer forces applied to the domain surface, i.e. the contact stresses.

The second term is the power of the outer forces applied to the domain volume, i.e. the remote forces.

Finally the last term, which is inferred from the previous ones, is connected to the deformation rate tensor. It is the power of the inner forces also called deformation power. We thus get the kinetic energy theorem statement: the differential with respect to time of the kinetic energy of a domain *D* followed in its movement is equal to the sum of the mechanical power of the outer forces and the mechanical power of the inner forces.

7.

In a few cases, we will encounter coupling problems between mechanical phenomena and thermal phenomena; therefore it is important to take a closer look at the contribution of thermodynamics.

Let us first address the first principle which establishes the equivalence between the mechanical energy and the thermal energy. The variation of the total energy, internal energy and kinetic energy, of a domain is equal to the sum of the power of the outer forces developed on the system and the heat quantity provided to the system per time unit.

The internal energy is an extensive magnitude which can be defined from the mass internal energy.

We saw earlier on that the same applies to the kinetic energy. In both cases we have to deal with state functions, i.e. functions depending on the configuration of our domain only.

In the other member we can find the power of the forces outside the domain.

It is completed by the variation of heat quantity that the domain exchanges with the outside. The last two quantities are not state functions and their values depend for instance from the variation path of our domain.

8.

With regard to the heat exchanged with the outside, we will find two contributions: one is the surface contribution, the other, the volume contribution.

The surface contribution term is connected to the conduction phenomenon. It is characterized by the flux of a common heat vector going through the surface bounding the domain. This conduction is a direct function of the temperature difference between the domain surface and the outside medium.

The volumetric contribution is connected to the radiation phenomenon defined by a heat rate. It is a function of the phenomenon involved: outer radiation, joule effect, internal chemical reaction or any other.

9.

The energy conservation equation clearly shows surface terms and volume terms.

It is then possible to take into account the kinetic energy theorem:

This gives a global form of the first principle of thermodynamics.

Using the continuity equation and applying the divergence and zero integral theorems, we can give a local expression of this balance. We can then say that the mass internal energy variation results from the mass power dissipated by the internal forces and a heat input.

10.

The second principle of thermodynamics does not express a conservation law, but the irreversible deterioration of a state function called entropy. The latter can only increase over time. It can take an extensive form connected to a volumetric distribution.

We can also give a global expression and a local expression.

Besides we have a fundamental equality:

Using this relation and the first principle of thermodynamics, we get Clausius Duhesme’s inequality.

11.

It is possible to introduce the free mass energy

We then get an inequality which shows the causes of irreversibility of a transformation.

The first two terms express the mechanical contribution with the deformation energy less its reversible part.

The last term involves the thermal contribution. The second principle of thermodynamics states that in the absence of any mechanical input, the thermal exchange between the two domains can only occur from the hot body to the cold body.

12.
Taking into account the notions of thermodynamics introduces new unknowns and it becomes necessary to use the the equations required to find these unknowns. In concrete terms, this means that it is necessary to give the temperature distribution law within the domain studied. It is the purpose of the heat equation.

In order to develop it, we assume that the material has perfect connections and there is no chemical or physical transformation. It is then possible to say that the variation of internal energy is proportional to the temperature variation, the proportionality coefficient being the mass heat capacity.

Besides, the volumetric density resulting from radiation is neglected.

The first principle of thermodynamics takes a slightly simplified expression.

With regard to conduction input, we can consider that the heat flux is proportional to the thermal gradient. This is Fourier law. The proportionality term may be tensorial if the domain is not isotropic.

We then get the expression of the heat equation. It is a differential equation requiring the definition of initial conditions and boundary conditions applied to our domain.

We may have a temperature prescribed to the domain surface.

But it is also possible to prescribe a heat flux, the latter being zero in the case of a perfectly insulated partition, i.e. adiabatic.

We can also prescribe a law for the convection heat exchange.

The same applies to the radiation heat exchange.

13.
Therefore we have seen that there is a deficit of equations with respect to the number of unknowns to find, which might lead to the fact that the number of solutions is undetermined. But reality is completely different and the solution often seems to be unique. There are complementary relations therefore making it possible to get rid of this ambiguity. These relations constitute the behavior law which depends essentially from the material. Formulating a behavior law depends on the assumptions made to obtain it which have to be in agreement with experimental results of course. Let us examine the specific case of the simplest behavior, i.e. linear elasticity.

The notion of elasticity can be expressed by the fact that the material response is described unambiguously by the knowledge of the temperature and the deformation tensor, which affects the free mass energy.

By transfer into Clausius – Duhem’s inequality, we get:

Since this inequality has to be satisfied whatever the temperature and the deformation tensor, we get two relations. The first relation shows that the free mass energy is the connection between the deformation state and the stress state. It is called elastic potential energy.

If we add to this the small disturbance assumption, by making a development limited to the first order, we get the expression of the linear thermo elasticity behavior law. The three terms of this expression can be analyzed independently.

The first term represents the stress state in the absence of mechanical or thermal stress. It is called residual stresses.

The second term shows a fourth order tensor, called elasticity tensor, doubly strained with the deformation tensor. This is the mechanical contribution to the stress state.

Finally the last term is the thermal contribution. It depends on the temperature difference with the initial configuration and a second order tensor representing the thermal dilation coefficients.

If we continue with simplifying assumptions, we can consider that the material is homogeneous and above all isotropic. The latter property implies that the behavior is identical in all the space directions. For this purpose the thermal dilation coefficient tensor is spherical, characterized by a single function, whereas the elasticity tensor is determined by two constants.

We then get a simplified expression of our behavior law, which constitutes Lamé's law.

From there it is possible to get the inverse formulation making it possible to calculate the deformation state since we know the stress state. This is Hooke's law.

These two new relations show four constants, however two of them only are independent in actual fact. For instance we have direct connections between Lamé's coefficients on the one hand, Young's modulus and Poisson's ratio on the other hand. The same applies to thermal contribution.

14.

Let us examine how we can find this result by gradually considering the assumptions required. We have to deal with a continuous medium in continuous and infinitesimal geometric transformation. The deformation state is characterized by the deformation tensor in small disturbances.

We assume that the transformation is mono thermal reversible. Reversibility implies that the second principle becomes an equality.

Indicating that the domain undergoes no chemical transformation and state change, we can infer that there is no heat produced internally. The domain being constantly at the temperature of its environment, the heat exchange is zero.

As a matter of fact, the equation expressing the first principle of thermodynamics has only one term in the second member. Which shows that the deformation power is the potential energy differential with respect to time.

This makes it possible to define the deformation energy which is the opposite of the work done by the internal forces. Using the first principle of thermodynamics, it is now possible to assert that this deformation energy is a state function. This deformation energy therefore only depends on the initial and final states, which implies that we have to deal with a total accurate differential equation. The material returning completely the energy supplied to deform it, the behavior is reversible, i.e. elastic.

To ensure that the deformation energy is a total accurate differential equation, the stress tensor components have to be functions of the deformation tensor only. They take a form defined within a constant which represents the stress state in the absence of deformation. They are the initial stresses or residual stresses.

Now we will assume that the undeformed initial state is natural, which implies that the residual stresses are zero. On the other hand we assume that the material has a linear elastic behavior, which implies that the stress state is connected to the deformation state by a fourth order linear tensor called the stiffness tensor.

Given the symmetry of the stress tensor and deformation tensor, this stiffness tensor is characterized by 36 terms. However we have fifteen relations between its components expressing Cauchy integrability conditions for the total accurate differential representing the deformation energy. Therefore we have 21 functions to determine. Generally speaking, these functions are time and temperature dependent, but by limiting the ranges of these two parameters, we assume that they are constants. These elastic coefficients are determined by behavior tests such as the tensile test.

15.

In order to reduce the number of coefficients, we can address the anisotropy assumption; but in order to understand the effects of this assumption, we will use a new form of writing for the elasticity tensor is a 4th order tensor which is difficult to represent.

To do so, we will use a linear application of our three-dimensional vectorial space in which the tensor of order two is represented by a matrix toward a six-dimensional vectorial space in which the symmetric tensor is represented by a vector. This is the transformation rule of the stress tensor.

With regard to the linear transformation applied to the deformation tensor, we use a slightly different transformation, which will be useful in the sequel to explain the deformation energy.

This makes it possible then to transform the fourth order elasticity tensor and represent it by a square matrix of dimension 6. It is therefore characterized by 36 terms.

Cauchy integrability conditions generate 15 relations between these various coefficients.

We have 21 independent coefficients. Given the integrability relations, the structure of the matrix representing the behavior is made up of two square matrices of dimension three and another in duplicate.

16.

The anisotropy conditions are linked to orientation changes in the material. In order to highlight the relations expressing symmetries for instance, we will use the basis changes. They are given by classical formulae for basis vectors:

We can then infer basis change relations for a second order tensor:

And we get similar formulae, even though slightly more complex, for a fourth order tensor:

17.

Let us examine what happens in the application of these formulae in the case of a simple plane symmetry first:

In this case the behavior law has to be invariant by means of the basis change defined by the matrices

Applying these basis change formulae then gives relations for the elasticity tensor coefficients:

This gives a new formulation of the deformation stress relations then.

18.

A medium is said to be **orthotropic** for a given property if this property is invariant by direction change obtained by symmetry related to two orthogonal planes, which automatically implies a symmetry with respect to the third plane. With the previous result, the matrix representing the elasticity tensor takes a very simple form in the orthotropic system.

If the orthotropic system is that of the deformation state Eigen vectors, the calculation result shows that we also get the components of the stress tensor in the main basis. Both deformation and stress main bases are then merged with the orthotropic basis.

The matrix expressing the elasticity tensor is characterized by 9 independent coefficients. Three longitudinal elasticity modules $E_1, E_2 \text{et} E_3$ in the orthotropic directions, three shear modules $G_{12}, G_{23} \text{et} G_{31}$ and three contraction coefficients $\nu_{12}, \nu_{23} \text{et} \nu_{31}$.

Indeed the symmetry of the first one in the matrix of dimension three gives three relations:

Finally it should be noted that there are thermodynamic considerations about the deformation work giving inequalities.

19.

A medium is said to be **isotropic transverse** for a given property if this property is invariant following a change of direction achieved by the rotation around a privileged axis. In that case, any plane going through the privileged axis is a symmetry plane. Therefore we can note that the medium is already orthotropic.

In the case of transverse isotropy carried by the third axis of the basis, it is necessary to have invariance of the behavior law for any rotation defined by:

With regard to the orthotropic case however, we will have additional relations between the elasticity tensor coefficients such as the ones suggested hereafter, for instance:

Finally we will have four new equations:

We then get a simplified matrix formulation in the transverse isotropy basis. Of course, if the transverse isotropy basis is the deformation state main basis, it is also the stress state main basis

20.

If the transverse isotropy rule is true for three orthogonal axes, the medium is said to be isotropic. We then get a set of complementary relations

The **isotropy** assumption prescribes that the behavior law is independent from the system selected to express it. In other words, the stiffness tensor has to be invariant for any basis change. Given all these relations, the behavior law writes simply:

The behavior law showing Lamé’s coefficients can be inferred. The main stress directions are also the main deformation directions. We will proceed with the applications using this behavior law.