STRESSES

1.
As will be seen later on, the kinematic approach of a study domain cannot be disconnected from a static approach leading to the transmission of stresses inside this domain. We will thus discover the notion of stresses which will be very useful to dimension structures correctly.

2.
The variation of a continuous medium over time cannot be random. It is directed by various laws or principles of physics.

The fundamental principle of mechanics is inescapable in a study combining forces and displacements. But one must not lose sight of the first principle of thermodynamics which permits to link the notion of force's working to that of heat by means of energy in all of its forms.

The second principle, represented by an inequality reflecting the irreversibility of any variation, is also an unavoidable classic.

But there are other relations, which are neglected sometimes but ultimately indispensable, like for instance the continuity equation translating mass conservation.

3.
All these principles are written in the form of laws representing the conservation of certain physical entities.

Writing these laws leads to equations which, according to the variables used, will take one form or another.

They are often presented in a global form for a complete material domain.

But they can also take a local form on an infinitesimal domain.

4.
In order to know the variations of our different quantities over time, we will isolate a domain at a given instant.

The quantity to study being distributed in the volume, it is defined by a volumetric integral.

We then have to examine what is happening to this quantity when the domain has changed.

5.
In the domain process of evolution over a very short time interval, we can see that there is a very significant overlapping domain.

Between the two moments, we can see both a loss and increase in volume.

6.
In order to calculate the variation between the two moments, we will calculate the integrals at each moment by decomposing systematically the integration domain into two parts.

Calculating the integral difference leads to three integration domains. We will then see that we make two contributions of a different nature. In the common domain, the integral as a function of time is important whereas the other two integrals reflect the fact that the integration volume varies over time.

7.

For the last two integrals the calculation of the element volume can be done considering that it is generated by the displacement between both instants of a surface element on the initial domain envelope.

This displacement is obtained by multiplying the velocity vector of the surface element central point by the time interval.

We then get an elementary cylinder.

Its volume is calculated by multiplying its height by its base area. With regard to height, since the displacement is not necessarily perpendicular to the base, it is to be calculated as the result of the scalar product of the displacement vector with the normal vector at the surface of the point considered.

This calculation is valid for the increase in volume, but for the loss, due to the fact that the scalar product gives a negative result, you have to take its opposite to obtain the element volume.

8.

The time interval being infinitely short, we then get:

The combination of the surfaces generating the increase in volume and the loss in volume giving the closed envelope surface of the initial domain, we can assemble the last two integrals together, which then gives the formula translating the particle differential of an integral. It is to be noted that the integrating function can be general, scalar vectorial or tensorial. We clearly see the appearance of the two contributions. On the one hand the temporal variation of the integrating function, on the other hand the domain spatial variation. It should be noted that the latter contribution can be interpreted as the flux through a closed surface.

9.

This notion of flux through a closed surface leads us quite naturally to mention the divergence theorem, or Green Ostrogradski theorem, which reads as follows: the flux of a tensorial field through a surface closing a domain is equal to the integral of the tensorial field divergence on the domain. It is a conservation theorem for it indicates that a domain inputs or outputs contribute to the storing variation of the entity in the domain.

In cases where the entity studied is a scalar function, the theorem takes a slightly different form:

This theorem being valid whatever the tensorial entity, it can apply to the resultant calculation of the distribution of constant pressure over a domain.

This pressure being constant, the connected gradient vector is zero and we can see that the resultant sought is zero whatever the form of the domain studied.

10.

Using this digression on the world of mathematics, we will enunciate the zero integral theorem. If, whatever a subdomain taken inside a larger domain, the integral of a tensorial field is zero, then this tensorial field is zero over the whole large domain. Proof by reduction to the absurd is possible by considering that there is at least one point inside the large domain for which the function is non zero. By continuity, it will be non zero over a domain enveloping the point and the integral over this domain will not be zero.

11.

A conservation law states the balance of a magnitude connected to a domain. We get an equation translating the fact that magnitude variation inside the domain is equal to the sum of the quantity produced inside the domain and the quantity going through the domain envelope surface. We get a global expression over a domain; however with the theorems mentioned above, it is possible to get a local expression valid at any point of the domain.

12.

The various formulations of the mass conservation equation, also called continuity equation, are an excellent illustration of the global and local expression notions. The mass conservation principle postulates that the mass variation of a domain over time is zero.

Using the volumetric mass, we move to a volume integral.

It is then possible to use the particle differential formula.

The divergence theorem permits to transform the domain surface integral into a volume integral.

We then get a first local form of the continuity equation.

It is also possible to use the following relations:

This gives a new local form of the continuity equation.

13.

There are several formulations making it possible to introduce mechanics. Depending on the presentation selected, what can be considered to be an axiom may become a theorem. The formulation we have selected, which is most commonly used as a first presentation, will subsequently result in demonstrating the virtual power theorem. However it should be noted that that we use the differential with respect to time of the Galilean kinetic torsor, not the Galilean dynamic torsor. The difference between these two torsors exists when the domain can lose mass, as is the case of rockets gradually transforming matter into energy.

14.

The nature of the forces applied by the external environment to the domain studied should be specified. They fall into two categories.

There are actions distributed in the volume and remotely applied. They include the gravity action or electromagnetic forces. They are represented by a force mass density defined at each point of the domain

Then there are actions distributed on the surface bounding the domain. They include for instance the pressure of the fluid enveloping the domain or the forces generated by the kinetic connections. They are represented by a force surface density defined at each point of the surface bounding the domain.

15.

Therefore the set of stresses applied to our domain permits to satisfy the fundamental principle of mechanics.

However if we divide the domain into two parts and isolate one of these parts, the forces mentioned earlier on do not necessarily permit to satisfy the fundamental principle of mechanics.

It is therefore appropriate to envisage that there are forces applied by the part removed to the part left. It is possible to envisage remote actions, but we will consider that they are surface actions generated by the breaking of interatomic bonds. They are represented by a force surface density called stress vector at point M in the direction of the cutting surface normal. This surface density is a function of the domain outer normal for the point studied. This is Cauchy's postulate.

16.
It is then possible to define the normal stress σ_n as the projection on the direction of the normal $\vec{n}$ of the stress vector $\vec{T}(M,\vec{n})$.

Similarly we get the tangent stress vector $\vec{\tau}_n$ (also called shear or shear stress) which represents the stress vector projected into the facet plane.

These components can be obtained by means of simple relations. A positive normal stress represents a tensile state of matter locally. If on the contrary it is negative, we get a compression state locally.

17.
The stress vector components represent a force related to a surface. They are therefore homogeneous at a pressure and will be measured in Pascal or mega Pascal, more generally speaking.

The stress vector thus determined applies to the structure in its actual configuration, i.e. deformed. It is the Eulerian version of this vector.

We have chosen an axiomatic system favoring the application of an infinitesimal force to the contact surface element. This results in problems of result interpretation in the presence of strong magnetic fields. In that case it will be necessary to consider applying an infinitesimal moment to the contact element.

If we had isolated the other part of our domain, we would have found opposed forces as a result of the mutual action principle.

18.
It must be clearly understood that the stress vector depends mainly on the separation plane of the two parts, a plane oriented by a normal vector. To realize this, we will address a cylindrical domain on the two bases of which uniformly distributed and opposed forces are applied. This domain is motionless and considered to be in equilibrium. We get a tensile load. Suppose you want to divide the domain into two equal parts. The first cut will be performed by a straight section plane, the second one by a meridian plane.

These two planes include the domain central point which will be our study point.

For the first cut, it is necessary to envisage a distribution of non-zero stress vectors in the cutting plane if we want the isolated domain to be in equilibrium yet.

On the contrary, for the second cut, we realize that our domain is in equilibrium without adding a stress vector. Thus, at each study point, there are as many stress vectors as cutting planes envisaged. This infinity of vectors raises questions and we will try to define the stress state at a point using another notion.

19.
To do so, we will establish the relationship between the stress vector and the normal vector connected to the cutting plane. Inside a domain in equilibrium, we isolate an infinitesimal tetrahedron three faces of which are perpendicular to the three vectors of an orthonormal basis.

Consider the face orthonormal to the first basis vector.

The stresses applied from outside to this face lead to the appearance of the stress vector.

To this vector it is possible to connect its normal component and its tangent component. The latter is included in the study face.

It is then possible to represent this vector by its components in the basis. These components will write with a natural index notation. The first index is the one connected to the study face normal and the second index is that of the projection vector. When both indexes are merged, we get a normal stress.

20.
A similar result is found on the second face. The normal component of the stress vector is carried by the second basis vector.

21.
The third face gives a similar result of course. We get a set of nine components to represent the stress vectors on these three faces.

22.
We still have to consider the fourth face oriented by a unit vector non co-linear to the basis vectors. It is possible to connect a specific stress vector to it.

23.
In order to obtain the forces connected to these faces, it is necessary to multiply these stress vectors by the areas of the surfaces to which they are applied.

First we should make a comment from the expression resulting from the divergence theorem applied to a scalar function.

If this function is constant, we get a very interesting result on the surface enveloping a domain. By adding all the vectors connected to the surface elements of a closed domain, we get a zero vector.

24.
Consider the application of this result to our element tetrahedral volume. The surface being elementary, it is possible to delete the integration sign. With implicit notations, our equation is written as the contributions of the different surfaces appear.

By projection to the basis vectors, it is then possible to deduce an interesting a relation.

25.
Our domain being infinitesimal, we neglect in the expression of the fundamental principle of mechanics all the volumetric contributions of a higher order than the surface contributions. On the other hand, it is also possible to delete the integration signs.

Using the relation we have just demonstrated, we can get a simplified relation:

And the mutual action principle gives a very interesting relation. With this, we can see that, from the knowledge of the stress vectors connected to the three vectors in a basis, it is possible to calculate the stress vector in any direction.

In a Cartesian basis, the stress vector components in any direction are deduced by means of a matrix relation.

We deduce an index formation from it.

This can be expressed by a tensorial relation showing the stress tensor, or Cauchy’s tensor. It is a function of the study point.

26.

This stress tensor makes it possible to know the stress distribution inside our domain. It depends on the loads applied, but also the kinematics of the medium studied. The dynamics equations will permit to write the dependence relation.

In order to write the fundamental principle of dynamics, it is necessary to express the torsor of forces applied from the outside. As mentioned above, we will have two contributions.

The first one is related to remote actions. They are distributed throughout the volume and represented by a mass distribution.

The second contribution is that of contact actions. They are distributed over the whole or part of the surface bounding the domain and are represented by a surface distribution which is actually the stress vector.

In the other equality member, we have the variation over time of the Galilean kinetic vector.

If we work with a conservative mass system, in Lagrange variables, it is equivalent to the Galilean dynamic torsor.

27.

Applying the fundamental principle of mechanics ensures a connection between all these torsors. For the moment we have a global expression valid for any material domain. We note that what we have is an expression in the form of a balance law. For two torsors, integration occurs over the whole domain, but for the contact outer force torsor, integration occurs over the closed surface enveloping the domain. Using the divergence theorem will permit to move to an integration over the whole domain.

To do so, we will use the connection between the stress vector and the stress tensor on the resultant.

The second member representing the stress tensor flux through our closed surface, applying the divergence theorem makes it possible to move to a volumetric integral.

Our resultant equation then takes a form with three integrals over the same domain.

By moving from mass integrals to volumetric integrals and using the zero integral theorem, we get a simplified expression. This equation is valid at any point of our domain. It represents the resultant equation of the Fundamental Principle of Mechanics.

28.
The equation obtained is essential to our mechanics course. It shows the connection between the stress tensor, the remote actions applied from outside to our domain and our domain movement.

In order to persuade the skeptical listener about the demonstration using the divergence theorem, we will repeat this demonstration isolating a parallelepipedical domain.

We position a system with respect to the domain, the system origin being merged with a top, the edges defining the axes. The domain is infinitely small. It is therefore bounded by six rectangular surfaces parallel to each other two by two.

For each face, it is possible to define the normal outside the domain,

The surface center position,

Determined by the coordinates in the system

The surface area,

29.
A stress vector is applied to the face studied. It is obtained by the product of the stress tensor defined in the surface center by the surface normal vector.

The first component of this stress tensor is obtained taking account of the function increases at the point of origin, increases resulting from the positioning differences of the two points.

We get a similar result for all the other components

30.
If we consider what is happening to each surface of the domain, we get the following table. The keen student may complete the table by representing the other five components of the stress tensor. Using this table, if we multiply inside a column the area by the stress vector component, we get the force component connected. Then we have to add all these force components to obtain the resultant of the surface forces applied to our domain. As can be seen, we will get many simplifications.

31.
Considering the projection to the first vector in our basis, we get a simple result for the resultant generated by the contact outer actions. The domain being infinitesimal, the integration can be neglected.

Calculating the projection of the resultant generated by the remote outer actions is simple.

And the same applies to the calculation of the projection of the Galilean dynamic resultant

Applying the Fundamental Principle of Mechanics then gives a scalar equation and obviously, the calculations of the projections to the other two axes of the system give similar results

This is expressed by the resultant local equation of the Fundamental Principle of Mechanics

The advantage of this demonstration lies in the fact that it gives the result of the moment equation of the Fundamental Principle of Mechanics rapidly. As a matter of fact, by writing this equation in the center of our domain, since many forces pass through this point, we get many simplifications and can see that the moment equation will apply on condition that the stress tensor is symmetric, i.e. equal to its transpose.

Thus, like the deformation tensor, the stress tensor will be represented by six independent components. These two equations represent the fundamental principle of mechanics locally.

32.

The relation between the stress vector and the stress tensor shows that we have to do with a linear application independent from the basis in which it is expressed. It is the characteristic property of tensors...

If we do not change a tensorial state when we change bases, it can be noted however that the stress tensor components are closely connected to the representation basis.

In another basis, it is possible to use the same type of formula to express a common component of the stress tensor

It is then possible to use the basis change formulae for unit vectors

This allows us to show the basis change matrix with its specific properties

We can ultimately get a general formula making it possible to calculate the components in any basis, only if they are known in one basis at least.

33.

Among all the bases, one at least is specific. The stress vector connected to a vector in this basis is co-linear to the initial vector; the co-linearity coefficient is called main stress.

Calculating these main stresses, which are nothing but the stress tensor eigenvalues, requires canceling a determinant

In the Eigen vector basis $(\vec{E}_I, \vec{E}_{II}, \vec{E}_{III})$, the matrix representing the stress state is diagonal. The tangent components are zero

The stress tensor being symmetric, it is possible to demonstrate this relation, called Cauchy's equality, which is valid with any unit vector.

By applying the previous relation with two Eigen vectors, it can be seen that, if the eigenvalues are different, the Eigen vectors connected are necessarily orthogonal.

34.

As with the deformation tensor, the decomposition into spherical and deviatory parts is unique when we express that the spherical tensor is proportional to the identity tensor and the deviatory tensor has a zero trace.

The opposite of the proportionality coefficient between the spherical tensor and the identity tensor is called the pressure. This notion is obviously very useful with fluids, but also with solids, especially in a plastic behavior.

35.
We have seen that calculating the eigenvalues requires canceling a determinant.

However, since the result must always be the same whatever the basis initially selected, the polynomial will be identical. The coefficients of this polynomial are therefore invariants for any basis change. The first invariant is the trace of the stress tensor whereas the third one is its determinant.

It is also possible to find index expressions valid in all bases.

In the case of the stress deviatory tensor, the first invariant is obviously zero. In the sequel we will see that its second invariant plays a specific role.

36.
For the stress tensor, it is obviously possible to use graphic representations as for the deformation tensor. The first representation is Lamé’s tridimensional ellipsoid.
We will address this representation beginning by designing an orthonormal basis. We assume that this basis is the basis of the Eigen vectors of a stress state whose main stresses are known. In this system, it is possible to represent a unit vector whose directional cosines are known. It is also possible to position the stress vector connected to this unit vector. All its components are known. The end of this vector is on Lamé’s ellipsoid. The Eigen vectors of the stress state also have the main directions of the ellipsoid. The intersection of this ellipsoid with the main planes gives three ellipses. Let us take a closer look at this distribution. When the unit vector varies, the stress vector varies too. In particular when the unit vector belongs to a main plane, the stress vector connected is in this plane as well. Its end is on the ellipse connected to the plane. And when the unit vector is merged with an Eigen vector, the stress vector is co-linear to this Eigen vector. It is also to take a look perpendicularly to the main plane. This permits to follow the full-size variation.

37.
And obviously it is possible to use the plane representation of Mohr’s circle too.

It is obtained by positioning three tangent circles. The diameters are equal unlike two eigenvalues and the centers are on the same axis.

Indeed the circle intersections with this axis give the main stress values.

For any unit vector, the end of the stress vector connected is inside the tri-circle.

Mohr’s representation permits to get directly the full-scale normal stress and tangent stress vector. Of course it is possible to get these entities by means of formulae.

By convention, the main stresses are ordered from the highest to the smallest.

The highest main stress is therefore the maximum normal stress.

The radius of the largest of the three circles gives the maximum tangent stress module.

38.
As an application we will now study a straight beam of circular cross-section.

A Cartesian system of coordinates is connected to it.

We will first consider that the beam is subjected to torsion. We will assume that at any point of the beam, in a natural cylindrical basis, the stress tensor only has a shear component.

On the other hand, we apply a pure bend load. This bend taken separately gives a stress tensor characterized by a normal component.

By applying the superposition principle, the stress tensor for the combined torsion flexion load is obtained by adding term by term the components of the simple load tensors.

39.
The structure of this stress tensor is specific. It is actually easy to check that the main axis of the cylindrical system is a stress main axis, the main stress connected being zero.

It can be deduced that the vectorial plane including the axial vector and the orthoradial vector is a main plane. There is therefore one of Mohr’s three circles which is connected to this plane. We will draw this circle by defining some of its points representing the end of the image stress vector of a unit vector taken in the plane.

40.
Let us calculate for instance the image stress vector of the axial vector in our basis.

We will represent this vector in Mohr’s plane...

To do so, we calculate the vector normal component

This component is on the normal axis carried by the axial vector

After removing the normal component from the image vector, there is the tangent component left

In this case, it is carried by the orthoradial vector

It is then possible to represent the image vector of the axial vector, which gives a first point located on Mohr’s circle connected to the main plane.

41.
In order to find a second point of the circle, we will use the orthoradial vector and the image vector connected to it.

The normal component is zero

It is logically carried by Mohr’s plane normal axis which, in this case, is the orthoradial vector

As a consequence, the image vector is purely tangential, the tangent vector being co-linear to the axial vector.

However we have to respect the orientation between the axial vector and the orthoradial vector we took for the first image vector.

It is therefore possible to represent the image vector. This gives a second point of Mohr’s circle.

It is therefore possible to draw the circle knowing that its center is at the intersection of the perpendicular bisector of the two points and Mohr’s plane horizontal axis.

The intersection of the circle with this horizontal axis gives two eigenvalues of opposite signs. We know that the third eigenvalue, connected to the Eigen vector merged with the radial direction, is zero.

This permits to represent Mohr’s other two circles.

42.

For the calculation of extremal eigenvalues, we have to work from the largest circle.

By construction, we know the abscissa of the center.

The Pythagorean Theorem permits to calculate the radius easily.

And we can deduce the eigenvalues.

And we know the stress state in the main basis

The method used to get this result is graphic, but it is possible to use an analytical method, of course. With this simple application, it can be noted that the tangent stress resulting from torsion contributes to increasing the maximum normal stress, just as the bending normal stress increases the maximum tangent stress.