DEFORMATIONS

1.

Continuum mechanics can be addressed either from the kinematic aspect, with displacements and deformations, or from the static aspect, with forces and stresses. As will be seen hereafter, although they are independent, these two aspects have a great deal in common and great duality. However, since the kinematic aspect allows addressing the notion of tensors in a way that is more physical, hence more easily understood by the student, our presentation will start with this approach.

2.

Being skilled mechanics, we will take from our toolbox the elements required to complete our task successfully. There are not that many.

We will need three nails in order to position correctly the points studied in our domain.

And in order to drive them in, we will take a hammer. However, as a precaution for our fingers, the nails will be sufficiently large and the hammer relatively small.

Finally we need two rubber bands. They will be stretched between two nails. But be careful, these rubber bands have very specific properties. They have to be infinitely small and infinitely stretchy. In order to reassure the students, it is worth mentioning that all these tools, hammer included, can be totally virtual. We simply need to use our imagination.

3.

We will try to find connections between the notions of displacement and deformations.

First it is important to note that it is quite possible to move a material field without causing deformations inside this field. The mechanics of rigid bodies is there to remind us of that. The distance between two points being a constant, the velocity field is a torsor field. And in the case of very small displacements, we also obtain a torsor field for the displacements; this field is characterized by a resultant and a momentum at a point.

It can be noted that the domain may very well be a fluid field, for instance contained within a bucket following a rectilinear path at a constant speed. We have not created any deformation yet.

It is therefore necessary to find inside the displacement field the properties permitting to generate deformations. And the problem is complex, for, as will be seen hereafter, multiple deformations may occur at each point of space, while the motion vector is unique for each point. The notion of deformation cannot be represented simply by a vector and we must use a new concept, the tensor, to define it.

4.

In order to physically determine the deformation state at one point M_0, we will position it in the reference configuration.

Then we will consider inside the reference configuration again a new point P_0, infinitely close to the previous point.

Using our big hammer, we drive a nail into each of these points without hurting ourselves.

The nails will serve as attachments to a rubber band stretched between the two points M_0 and P_0.

We thus obtain the vector $\overrightarrow{dX} = \overrightarrow{M_0 P_0}$ which is defined in the reference configuration.

We then transform our domain prescribing a displacement field.

In the actual configuration, determined at instant *t*, the image points of the previous ones are to be found, i.e. points M_t and P_t.

With regard to point M_0, its motion vector is $\overrightarrow{U(M_0)} = \overrightarrow{M_0 M_t}$. The same applies to point P_0. $\overrightarrow{U(P_0)} = \overrightarrow{P_0 P_t}$

Thus, in the actual configuration, the position of our rubber band can be determined using vector $\overrightarrow{dx} = \overrightarrow{M_t P_t}$

5.

We now want to look for the connections between the initial vector $\overrightarrow{dX}$ taken in the reference configuration and its image by means of the motion field in the actual configuration, i.e. vector $\overrightarrow{dx}$.

To do so, we will simply differentiate in space the present position vector. Assuming that the latter is known by its components in a Cartesian coordinate system and that, in this case the basis vectors are invariants, the image vector will be simply obtained by differentiating the components

The latter depend on Lagrange variables and we get a spatial differential form which is given by considering the variations with respect to the initial position variables, i.e. by considering the differential of a function with several variables. By convention, the upper case indexes are connected to the reference configuration and the lower case indexes are connected to the actual configuration.

The latter index relation can be presented in a matrix form

6.

The matrix above represents in our basis the components of a tensor denoted **F** and called **Gradient tensor** or else **Tangent linear application**. It permits to calculate our image vector knowing the vector in the initial configuration.

It is also possible to start from the inverse relation.

And we can also connect this tensor to our motion field. To do so, it is sufficient to write that the motion vector is the difference between the position vector in the actual configuration and the position vector in the reference configuration. A new notion then appears: it is the gradient tensor of the motion vector.

7.

To make it more concrete, we will now see how we can calculate the components of a gradient tensor in the case of a very simple transformation: homogeneous tri-axial deformation.

For this deformation, a rectangular parallelepiped built according to the basis vector axes is transformed into a new rectangular parallelepiped.

8.

The equations of this transformation are very simple. They represent proportionality between the present coordinates and the reference coordinates.

Differentiating the present coordinates, it is possible to obtain the components of the gradient tensor. The latter takes a diagonal form.

Assuming that the reference configuration volume is a unit volume, the actual configuration volume is given by the product of the tensor values. It also corresponds to the matrix determinant. This result, taken from this example, can be generalized. The gradient tensor determinant gives the volume relative variation in our transformation.

9.

Let us now examine how we can determine the gradient tensor components in the case of a rigid solid motion. As seen earlier on, under the assumption of small displacements, the motion field, which is a torsor field, is determined by a resultant, the rotation vector $\overrightarrow{\theta_s}$, and the momentum at one point, i.e. the motion of this point $\overrightarrow{U(A_0)}$.

The components of the motion vector of any point are then given by a vector relation, which can be translated into an index relation:

In order to obtain the gradient tensor components, it is sufficient to use the index relations defined from the motion field:

Under the predominant assumption of small displacements, this tensor determinant is equal to 1, i.e. there is no volume variation.

10.

Using the previous example, we can see that the gradient tensor cannot be representative of the deformation state. For an undeformed solid it is not zero. To characterize the deformations, it is necessary to reflect the fact that the distances between two points and the angles around two directions will change during the transformation. To do so, we will use the scalar product of two vectors which is a function of both the norm of these vectors, hence the distances between two points, and the angle formed between these vectors.

Consider two vectors in the reference configuration.

To locate them, we hammer the nails at the vector extremities and stretch rubber bands between two nails.

We then prescribe the motion field to our domain.

The nails and rubber bands will then position themselves in the actual configuration.

The new vectors in the actual configuration are symbolized by the nail locations. The varying lengths of the rubber bands permit to visualize the elongations.

It is also possible to see the variation of the angle initially formed between the two vectors.

11.

With the gradient tensor we have seen that it is possible to calculate image vectors from vectors defined in the reference configuration.

The scalar product of two vectors is obtained in a matrix form by transposing the first vector representative.

Developing the calculation, a new matrix appears, representing a tensor called the right hand Cauchy Green Tensor. It is obtained by performing the product of the transposed gradient tensor by the gradient tensor.

12.

The developed form of the matrix calculation shows the necessity to transpose the first vector if one wants to obtain a scalar function.

The calculation of the index components shows that the tensor obtained is symmetric. On the other hand, it is Lagrangian, both indexes defining it being connected to the reference configuration.

It can be calculated directly from the motion field and the motion gradient tensor. Transposing the latter expression, the same result is obtained, which demonstrates once again the right hand Cauchy Green tensor symmetry.

13.

With all the tools required at our disposal, it is now possible to envisage the calculation of our scalar product during the transformation.

This gives the opportunity to highlight a new tensor called Green Lagrange tensor. It is also a Lagrangian symmetric tensor. Its components in a basis will be calculated from an index formulation.

14.

Consider again the example of the tri-axial homogeneous transformation for which we have already identified the gradient tensor.

Using the formula $\mathbf{C}={}^{T}\mathbf{F}\,\overline{\otimes}\,\mathbf{F}$, it is easy to calculate the components of the right hand Cauchy Green tensor, which is then a diagonal tensor.

Then the formula $\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{I})$ permits to obtain the components of the Green Lagrange tensor:

15.

All these calculations were made taking the initial configuration as reference configuration. This is the Lagrangian approach. But it is possible to use the actual configuration as reference configuration. We then switch to the Eulerian approach.

For the Eulerian observer, the reference scalar product is the one in the actual configuration and its variation is calculated during the transformation. The results are of the same form as in the Lagrangian approach. With the left hand Cauchy Green tensor, the scalar product can be calculated in the initial configuration.

Euler Almansi tensor, a Green Lagrange tensor dual, will permit to have the scalar product variation. The last two tensors are purely Eulerian. Like their counterparts in the Lagrangian approach, they are symmetric.

There exist bridges between these two approaches. The Eulerian approach is more commonly used for fluid fields, i.e. fields for which it is difficult to define an initial configuration. But we will see in the sequel that, by means of additional simplifying assumptions, the Eulerian and Lagrangian approaches will be merged.

16.
Using the previous tools, it is now possible to define measurable physical magnitudes.

Let us begin by positioning in the studied point reference configuration an infinitely small vector $\overrightarrow{dX}$ to which its length dX is connected.

Its image vector is defined by the motion field. It can be noted that its direction is not necessarily identical to the direction of the initial vector.

Linear dilation, or elongation, is defined as the initial vector relative variation in length. It depends on the point studied and the direction in which the measurement is made.

It can be calculated either from the right hand Cauchy Green tensor, or from the Green Lagrange tensor.

It is also possible to calculate the linear dilation in specific directions. It can be noted for instance that the first components of the right hand Cauchy Green and Green Lagrange tensors are connected to direction $\overrightarrow{E_1}$.

17.
Determining linear dilations is not sufficient to characterize the deformation state of a domain. One must also calculate angle variations.

To do so, we will consider two directions defined in the reference configuration we will follow during the transformation.

In order to establish an initial norm, we prescribe a right angle in the reference configuration to our two directions; we then note the variation of this angle in the actual configuration.

This variation then determines angular distortion or sliding. This angular distortion depends on the point studied and the orthogonal directions initially selected.

The scalar product of two vectors depending on the cosine of the angle formed between these two vectors, or the sine of the complementary angle, it is possible to envisage determining angular distortion calculating the scalar product of the vectors in the actual configuration.

The result can be connected either to the right hand Cauchy Green tensor or the Green Lagrange tensor.

And it can be seen, when taking vectors from our basis, that there is a direct connection with the components of the tensors.

18.
To illustrate the calculation of the Green Lagrange tensor components, consider once again the example of tri-axial homogeneous deformation.

19.

It consists of elongating or shortening a rectangular parallelepiped in the three directions formed by the edges. Consider a specific plane. In the reference configuration, we position at a common point a horizontal vector and another vertical vector. We then prescribe the deformation to our domain. The ends of our vector are fixed to points and follow the motion vectors of these points. In the actual configuration, our vectors show variations with respect to the reference configuration. By studying these variations, it is possible to measure the length relative variation on the one hand, and on the other hand the variation in the angle initially formed by our vectors, a variation which is actually zero. With these data, it is possible to infer certain components of our Green Lagrange tensor.

20.

Consider once again the previous example, but instead of working in directions parallel to the edges, we will work along directions angled at 45°. We will prescribe the same deformation to our domain. In our specific plane, the study axes will therefore be defined by a 45° rotation with respect to our previous axes. At a point studied, we indicate two orthogonal vectors in the reference configuration, and then we prescribe our deformation. It is easy to see that this time the angle formed between the two vectors is changing, while they are subjected to the same elongation. The Green Lagrange tensor components are no longer the same although the same deformation state is represented. In the basis with vectors parallel to the edges, the matrix representing the tensor was diagonal. It is said that we are in the deformation main directions.

21.

The tensors we have determined are symmetric with actual components. The matrix representatives can be diagonalized. This means that it is possible to connect a specific basis in which the matrix representative will be purely diagonal.

An elementary rectangular parallelepiped constructed in the reference configuration according to this basis will be a rectangular parallelepiped again in the actual configuration for the angular distortions are zero.

It is then possible to envisage calculating the volume relative variation.

It can be obtained easily from the right hand Cauchy Green tensor components.

This reveals the jacobian of our tensor.

22.

Formulating deformations occurring inside the material can therefore take a relatively complex form. With the expressions given above, we realize that the deformation state is not linear.

In other words, the total deformation resulting from a series of displacement fields is not the simple addition of the elementary deformations generated by each displacement field taken separately.

From a mathematical viewpoint, this non-linearity results from the product of the two tensors in the calculation of the Green Lagrange tensor.

However in many applications, this term is not predominant with respect to the two previous ones and we can neglect it. Thus our transformation has been linearized.

23.

In order to obtain this linearization, we will formulate the assumption of infinitesimal transformations, also known as the assumption of small disturbances. This primarily comes down to two hypotheses:

The displacement gradient tensor contains only small terms in front of the unit.

The displacement of each point of the material field is small. This last assertion is important as it implies that it is possible to merge the present state with the initial reference state. There is no more reason to distinguish between these two states in our notations and the upper case and lower case indexes become equivalent.

We then note that there is total coincidence between the Lagrangian and Eulerian approaches. The Green Lagrange and Euler Almansi tensors are identical. To avoid promoting one of these approaches, we will denote ε this tensor which will characterize our deformations.

In a Cartesian coordinate system, the calculation of these tensor components is very simple.

24.

With the previous relation, it is easy to see that the deformation tensor is actually the symmetric part of the displacement field gradient tensor. It is defined by six components.

It is also possible to define the anti-symmetric part which will give a tensor determined by three components only, the terms on the matrix representation diagonal being necessarily zero.

It is shown that it is possible to connect a vector to such a tensor so that the tensorial product is replaced with a vectorial product.

The gradient operator, which is actually a simple spatial differential operator, is linear.

The result is that the deformation tensor is linear too.

25.

It can be noted that the image vector of a vector in the reference configuration depends on the displacement field gradient tensor, hence its symmetric and anti-symmetric parts.

From the connection between these two vectors and using Chasles relation, we deduce a relation between the displacement vectors of two neighbor points in the reference configuration between which a rubber band has been stretched.

These two displacements are interconnected by means of symmetric and anti-symmetric tensors.

By decomposing the displacement of the second point, one can see that it occurs after the displacement of the first point.

Then a vector is connected to the anti-symmetric tensor; this vector can be obtained by simple rotation due to the fact that we are placed in the small disturbance assumption.

Finally a third component is connected to the symmetric tensor. By means of this decomposition, we can see that the rubber band initially stretched between points M_0 and M'_0 actually deformed over its last part only. Therefore it is logical to assert that symmetric tensor ε is the deformation tensor. If the latter is zero, locally the displacement field simply becomes a translation combined with a rotation, which generates no deformation.

26.

The small disturbance assumption is made for many applications; however some of them cannot accept it. And yet it actually simplifies calculations.

To prove this you only have to examine the case of a beam whose end is clamped, which supports a load uniformly distributed over its whole length. In beam theory, you work on the structure considering it in its reference configuration, i.e. as if it was not deformed. But, strictly speaking, if the deformations are somewhat large, therefore if the small disturbance assumption is no longer applicable, you have to make the calculations in the actual configuration, taking account of the deformation. Then the issue of the accurate definition of loading is raised.

Does the loading, like gravitational actions, keep a constant direction whatever the deformation of the structure?

Or does the loading, like pressure actions on a surface, have a direction related to the deformation of the structure?

27.

From the answer to this question, the expression of our flexion moment will result. As a matter of fact, in the first instance, the distance between the loading at the calculated point and the bending moment is calculated in horizontal projection only.

Whereas in the second instance, this distance, which must be considered perpendicularly to the loading line of action, is a function of the deformation. And then, according to the value of the bending moment, the structure deformation will follow. It is easy to see from this example that without the small disturbance assumption, the calculations will be complex somehow.

And this has a different meaning when you go into the calculations. The classical formula which permits to obtain the deformation equation of a bending beam leads to a second order differential equation. If you look closely at the demonstration of this relation, you can see that the beam slope was considered to be negligible.

If this is not the case, the real equation is far more complex. It certainly still gives a second order differential equation, but seeking a mathematical solution is far from obvious.

28.

With the small disturbance assumption, the terms inside the deformation tensor are negligible with respect to unit $\varepsilon_{ij} \langle\langle 1$

Due to this approximation, calculating the linear dilation in a direction linked to a vector in our basis takes a very simple form.

This also applies to the angular distortion of a right angle formed between two vectors in our basis. The two previous results show that the components of our deformation tensor can take significant physical values.

Therefore, in a data base, the terms on the tensor diagonal will be the linear dilations in the directions of the basis vectors. With regard to the terms outside the diagonal, we find the semi-angular distortions of the right angles formed by the vectors in our basis.

29.

But it is also possible to work with simple unit and orthogonal vectors.

The linear dilation in a unit direction is rapidly obtained from the knowledge of the deformation tensor.

Similarly, with the deformation tensor, it is simple to calculate the angular distortion for two orthogonal vectors. Given the symmetry of this tensor, it is possible to use two different expressions for this calculation.

Consider any unit vector: its image through the deformation tensor permits to determine a new vector which will be called the pure deformation vector at the point considered in the unit direction selected.

Finally the volume relative variation at the point considered actually leads to the trace of the deformation tensor, i.e. the displacement vector divergence.

30.
In the basis studied, the deformation tensor is represented by a symmetric matrix with actual coefficients. As seen above, it is possible to find a basis in which the tensor takes a purely diagonal form. This basis can be defined by the diagonalization process, i.e. by seeking Eigen vectors and Eigen vectors linked to our deformation state.

An Eigen vector is a vector co-linear with its image by means of our tensorial application, the co-linearity coefficient being the Eigen value.

Using the previous equation it is possible to deduce an important relation permitting to obtain the Eigen values

On the other hand, given the symmetry of our deformation tensor, it is possible to write the so-called Cauchy relation:

This shows that if two eigenvalue vectors are different, then the connected Eigen vectors are orthogonal.

The three Eigen vectors can therefore make up a direct orthonormal basis in which the matrix representative of our deformation state is purely diagonal. We call main directions the directions given by the Eigen vectors. The eigenvalues will be called main deformations.

31.
For any tensor, seeking the eigenvalues involves canceling the characteristic polynomial.

This polynomial can be defined from any matrix representative of the tensor, i.e. in any basis. However the roots of this polynomial, i.e. the eigenvalues, always have to be the same. As a consequence, the polynomial is independent from the basis.

This means that the polynomial coefficients are invariants by means of a basis change. The first invariant simply represents the trace of our matrix. The last one is the determinant. The second one is the half difference of the squared matrix trace with the matrix trace multiplied by itself.

By doing the calculation in the main basis, we obtain fairly simple results.

32.
For any second order tensor, it is always possible to decompose it in the form of two tensors so that one is spherical and the other has a zero trace. The decomposition is unique.

In the case of deformations, as the tensor trace represents the volume relative variation, with the spherical tensor, the volume changes without changing the form whereas with the deviatory tensor, the form changes without changing the volume.

Consider the example of a deformation prescribed to a field.

As we are under the small disturbance assumption and can linearize the deformations, it is quite possible to consider that we first have a deviatory state followed by a spherical state.

But it is equally possible to assume that we prescribe a spherical state followed by a deviatory state.

33.

In order to strengthen our knowledge of decomposition in a spherical and deviatory state, we will use a MECAGORA training project interactive module. This module uses the VIRTOOLS software. If it does not already run on the computer, it will automatically install upon your initial login.

The mouse operating instructions are available at all times by clicking on the Information menu.

A classical screen displays several windows.

A graphic visualization window contains 3D objects which can be moved interactively using the mouse.

A control window in which it is possible to modify parameter values.

And a result window containing the numerical values of a number of calculated magnitudes.

34.

In the IB4 module, after activation, you obtain in the graphic window a cube that can be deformed by prescribing new values in the control window. When prescribing a linear dilation in different directions, we can see in the result window that the spherical tensor changes and the deviatory tensor are diagonal. On the contrary, it is no longer so when angular distortions are prescribed to our initial cube.

35.

The graphic representation in the Mohr plane is very interesting. As a reminder, this plane is the vectorial plane containing the normal axis along which we want to perform the study at one point and the connected image vector. In the case of the deformation tensorial application, this vector is the pure deformation vector.

Projecting the pure deformation vector onto the normal axis gives directly the measurement of the linear dilation at the point studied, along the direction of the normal axis studied.

The projection onto the tangent axis makes it possible to have the half value of the angular distortion at the point studied, for the right angle initially formed between the normal axis and the orthogonal vector belonging to Mohr plane.

36.

In the representation of Mohr’s tri-circle, we know that the end of the image vector must be inside the area bounded by these three circles. Therefore the projections onto the basis vectors are equally limited.

37.

The largest main deformation represents the highest linear dilation at the point studied.

The smallest main deformation represents the lowest linear dilation at the point studied.

Finally the largest angular distortion is given by the diameter of the largest circle, i.e. the difference between the largest main deformation and the smallest one.

38.

The deformation tensors, i.e. second order tensors, can be calculated by differentiating the displacement field, i.e. a first order tensor.

The index formulae in Cartesian coordinates for the calculation of the deformation tensor six components show that they are simply deduced from the displacement vector three components.

It may be useful to envisage the reverse approach, i.e. to determine the displacement field connected to a deformation state. Such a transformation is not instant for it implies the integration process of six scalar functions to go back to three scalar functions constituting the components of a vector.

In order for this to occur, the deformation tensor components have to be interconnected. These connections are called the compatibility conditions.

39.

These conditions, which are only integrability conditions of accurate total differential equations, are obtained from the anti-symmetric tensor.

Carrying out the demonstration in a Cartesian coordinate system, it can be noted that the spatial differentials of the components of this anti-symmetric tensor are connected to the spatial differentials of the deformation tensor.

40.

As a matter of fact, these spatial differentials can be interpreted as the gradient vector components of this anti-symmetric tensor component.

But this will only be a gradient vector if the rotational connected to it is zero, which gives the following relations:

Which constitute actually the Cauchy integrability conditions of an accurate total differential.

According to the deformation tensor components and given its symmetry, we actually obtain a system of six independent equations.

41.

These equations can be presented in a developed form in a Cartesian coordinate system.

But the general form valid in any coordinate system is as follows:

42.

If the compatibility conditions are satisfied, it is then possible to determine the displacement fields generating the deformation state.

To do so, one first has to integrate the differentials giving the three components of the anti-symmetric tensor.

Then it is possible to go back to the three components of the displacement vector by integrating the differential forms connecting the displacement vector to the deformation and anti-symmetric tensors.

A keen student can apply the method in order to find the displacement fields generating the deformation tensor proposed.

43.

In the study above, we have addressed the system transformation between a reference configuration C_0 and an actual configuration C_t. Paying no attention to the deformation path followed when moving from one configuration to another, we have considered the transformation from a purely geometric aspect, from an initial state to a final state. It is easy to understand that this study is appropriate for any transformation in which the deformation state is a state function in the thermodynamical sense of the term. No matter the path from one state to another.

44.

Especially in plasticity processes, it is sometimes necessary to study the variations of the system following a deformation path. We then have to carry out the study in an incremental way, i.e. to study the transformation between two infinitely neighbor states, then, by means of an integration process, deduce the actual deformation path.

45.

Between two infinitely neighbor instants, we will address the variation velocity of the scalar product of two vectors.

Consider the Lagrangian approach first.

We obtain the Lagrangian deformation rate tensor which is actually the deformation variation velocity when it is measured from an initial reference state.

46.

In the Eulerian approach, the calculations are slightly more complex.

It is necessary to use the gradient tensor and its temporal differential

47.

The calculation of the scalar product variation has to be done with respect to the vectors expressed in the actual configuration.

This allows showing the Eulerian deformation rate tensor.

48.

But we can see that the product of the differentiated gradient tensor by the gradient tensor gives a tensor representing the velocity gradient.

Thus the Eulerian deformation rate tensor is actually the symmetric part of the velocity gradient tensor. It is also possible to show the anti-symmetric part which represents the material rotation velocity.

49.

We can see that the Lagrangian deformation rate tensor can be easily obtained by temporal differentiation of the Green Lagrange tensor and the Eulerian deformation rate tensor can then be obtained with the gradient tensor.

On the contrary, in the Eulerian approach, the differentiation calculations are more complex as shown by the formula giving the temporal differential of the left hand Cauchy Green tensor.

50.

One solution to describe a variation over time is to consider each calculation step as reference situation for the next calculation. The displacement being very short, the deformation tensor is linearized.

Within the framework of infinitesimal transformations, the Lagrangian and Eulerian deformation rate tensors can be merged.

51.

As with the Green Lagrange deformation tensor, the various components of the deformation rate tensor have physical meanings.
If in the actual configuration we take vectors co-linear to the first basis vector, we can see that the first component of the deformation rate tensor is the dilation rate in this direction.

If we take vectors in two orthogonal directions, we get the right angle slipping rate.