DESCRIPTIONS

1.

The objective of this course is to give a brief survey of continuum mechanics. We will confine ourselves to the linear elastic behavior field. We will first define the variables used to make a description of the mechanical state.

2.

The objective of Continuum Mechanics is to describe the mechanical aspect of material fields whatever the state of the material, solid – liquid – gas – plasma, making it up. The tools used for this description will therefore be identical to deal with deformable solid bodies and fluids. They also permit to describe state changes.

3.

Besides, even if physics leads us to distinguish the different states, the notion of state change can be questioned by considering the example of the water entropic diagram.

From a liquid solution, by means of isobar heating, it is possible to move to the vapor state going through a transition phase during which both phases are mixed

But if from the vaporization point at a temperature higher than that of the critical point, isothermal compression is performed followed by isentropic expansion, it is possible to return to the initial liquid state without ever feeling that there has been a phase change affected by the two-phase mixture.

This is an illustration of how difficult it is sometimes to distinguish the states of matter and fully justifies the fact that with continuum mechanics we will not try to give a mechanical description related to a specific state.

4.

And if one takes a close look at sand, can one say that it is solid or liquid?

What about yoghurt which, after stirring with a teaspoon, will go from the quasi-solid state to the liquid state?

Old stained-glass windows show, with the droplet shape one can see when looking at the thickness variation according to height, that glass, over a relatively long period of time, behaves like a fluid. It is true that, over a shorter period of time, it will be considered to be a very brittle solid.

Some silicones have very strange behaviors: they may run through the fingers and become brittle under stress.

Analyzing those various behaviors will lead us to rheology but time is short to explore all its complexities in the present course. It is easy to imagine that numerous parameters such as pressure, temperature and time for instance will be included in our descriptions. We will ensure to have a fairly simplified model to be comprehensible, but a model that can be used in everyday life however. It goes without saying that the simplifications appearing in the form of assumptions will involve differences between our theory and reality. Of course we will need to ensure that these differences are minimized.

5.

The problems we will have to deal with will be relatively complex and, as a consequence, it is very likely that we may not get satisfactory mathematical answers. In order to obtain successful calculations, we will have to make assumptions whose main objective is to simplify the mathematical models to deal with. We will make these assumptions gradually during the course. We will begin by specifying what is meant by continuity in continuum mechanics. First this continuity has to concern the field under study.

A field will be said to be continuous if all the functions characterizing it are continuous spatial functions, i.e. if they are indefinitely differentiable in space. Thus the volumetric mass, temperature, pressure of a fluid, the velocity of the points of a solid have to be functions with no discontinuity. Contrary to what one may think, it is not always easy to comply with this continuity condition.

There is obvious discontinuity in a field consisting of water and oil. It is evident that at the interface of both liquids, the volumetric mass is discontinuous. That being so, it will be possible to use continuum mechanics to deal with a water oil mixture problem. It is sufficient to treat separately water, considered to be continuous, and oil, a continuous fluid too, then put the pieces together and write the correct equations governing the interface between the two liquids. Of course this is much easier when the two liquids are miscible and form a continuous medium together.

But, although wine is miscible in water, the mixture obtained, despite the fact that it gives an impression of continuity, is no longer so at the atomic scale. Of course, at this scale, no field is continuous. But at the human scale, there is an impression of continuity. If we assume that the wine water mixture is continuous, we make a simplifying assumption making it possible to work with differentiable functions therefore facilitating the solution of our problem. The more approximate the assumption, the more differences are to be expected between our calculation results and reality.

If one considers that reinforced concrete is continuous, one can expect relatively incorrect results. However they will be relatively more correct in the Millau Viaduct pillar calculations than in apartment floor calculations. It is easy to understand that in large structures reinforced concrete is more continuous than in small ones.

6.

However spatial continuity is insufficient. It is necessary to have temporal continuity as well. The characteristic functions of our field have to be infinitely differentiable with respect to the time variable.

For instance the volumetric mass of a body being heated will be a continuous function over time. Again this time continuity is not always obvious.

Cavitation phenomena are examples of discontinuity. When in liquid water one goes to a point below the saturation vapor pressure, a vapor bubble is formed instantly. Of course the volumetric mass will change abruptly.

Similarly fracture mechanics whose objective is to study crack propagation is full of examples of discontinuous functions, such as the displacement of a point for instance. But here again, using gimmicks, we will deal with fracture mechanics taking continuum mechanics as a basis.

7.

Before defining the variables that can be used to make a study, it is appropriate to specify the way in which these variables will be identified. To do so we will define the notions of reference systems and coordinate systems.

The reference system is connected to the observer. It represents the set of mobile points in a rigid body in front of the observer. The functions studied are defined with respect to this reference system. For scalar functions such as temperature or volumetric mass, the reference system is sufficient to characterize them. When it comes to the temperature in the point of a body, this designates the value of a function whose deviation from a reference system is measured. On the contrary, for more complex functions such as vector functions like a point velocity, it is common to symbolize them with their components on coordinate systems.

For clarity and visualization, not necessarily for writing, we will therefore link coordinate systems to the reference systems used. The coordinate system, characterized by the selection of a so-called point of origin and a basis, will permit to describe the scalar functions which will be no less than the components of the vector function in the coordinate system. Of course it is possible to link several coordinate systems to a reference system. Thus, while keeping the same point of origin, it will be possible to talk about a Cartesian coordinate system or curvilinear system such as the cylindrical system. One must have a clear understanding that changing the coordinate system, i.e. the components studied, will not fundamentally modify the complex function studied. It is then called tensor function.

In the example illustrating the squeeze of a block between the plates of a press, it is possible to define two different reference systems. One is linked to the upper plate, the other to the lower plate. It is essential that the two observers of these reference systems represent the same physical phenomenon in their own reference systems and can exchange their data easily. This is the objectivity property of the phenomenon. The final state of the block has to be identical for the observer linked to the upper plate and for the observer linked to the lower plate.

8.

Once the reference system, and possibly the coordinate system, has been selected, it is possible to envisage the description of the variation of our study variables. Then we will see that there is the so called Lagrangian approach and the so called Eulerian approach. But first, let us provide some information on the index notation we will use hereafter.

Consider a reference system linked to a point of origin denoted O and a basis. The basis vectors will be denoted E1, E2 and E3 with indexes and not X, Y and Z, as is often the case.

The position of any point M in space is given by vector OM defined in our reference system. In order to have a representation of this vector, we will use the scalar components X1, X2 and X3 obtained by projecting the position vector onto the basis vectors.

Therefore we can write: $\overrightarrow{OM} = x_1 \overrightarrow{E_1} + x_2 \overrightarrow{E_2} + x_3 \overrightarrow{E_3}$.

This notation is rather long to write, so a shorter notation is generally preferred using the summation sign: $\overrightarrow{OM} = \sum_{i=1}^{3} x_i \overrightarrow{E_i}$. Index *i* then takes three values, which gives the three terms of our vector.

However, since we nearly systematically work in a vector space of dimension 3, a new writing simplification can be made by removing the summation sign: $\overrightarrow{OM} = x_i \overrightarrow{E_i}$. The expression then becomes very much reduced. It represents what is most necessary only. It is the Einstein notation system.

9.

The Einstein notation system makes extensive use of the index notation. The rule is relatively simple: if in a monomial, an index is doubled, then there is automatic summation for this index of all the values it can take... In our applications, since we mainly operate in a three-dimension vector space, it means that the doubled index can take values 1, 2 and 3. Such an index is called mute index. It may be replaced by any other letter, the result does not change. An index which is not repeated in the monomial is said to be an actual index or speaking index in contrast. An actual index has to be identical in both terms of an equality. It is necessarily unique. In the Einstein notation system, it is possible to find only doubled indexes, therefore mute, or unique indexes, therefore actual. It is impossible to find tripled or quadrupled indexes, for instance. In the examples given, the scalar product of two vectors is in a central position. The final expression represents one of the simple forms of Hooke's law. Connoisseurs will appreciate.

For purists, higher and lower positions should be used in monomials for mute indexes. These are referred to as contravariant and covariant indexes.

But we will not use this notation so as not to confuse the reader, even though it provides major advantages to those who can use it.

10.

We will now treat a few examples in order to get familiar with this notation system which will be regularly used in the course.

First consider the previous formula given for the scalar product of two indeterminate vectors. The result depends on the scalar product of the vectors making up our basis. In the general case, this basis will be orthonormal. The scalar product of two unit vectors is 0 if both vectors are orthogonal and 1 if both vectors are combined.

Kronecker's symbol will then be used to represent this result. In this symbol, if both indexes composing it are different, it equals 0 and if both indexes are identical, it equals 1.

But then, in the double summation representing the scalar product, it is useless to keep the terms for which both indexes are different.

Keeping only the terms with identical indexes for Kronecker's symbol, it is equal to 1. And the final result is very simple then.

11.

We can also consider the index expression of a number of operators commonly used in physics.

To do so, first consider a continuous scalar function $s(M) = s(x_i)$ defined at any point in space. It is assumed that the value of this function depends on the point position, a position given by variables xi. This will be referred to as a scalar field.

It is then possible to examine, via the gradient operator, the spatial variations of our function. We obtain a vector. In a Cartesian system of coordinates, the components of this vector are given by the partial derivatives of function s(xi) with respect to the position variables respectively.

It is also possible to define a vector continuous function V also defined at any point in space. This time the components in a Cartesian basis will depend on the point position. This will be referred to as a vector field.

We can give the expression of a vector divergence. We then get a scalar function whose value is the sum of the partial derivatives of the components with respect to the position variables respectively. From these results many others can be easily obtained.

For instance, if one wants to express the divergence of a function multiplied by a scalar function, it is sufficient to make the calculation with the index notation to obtain the result.

12.

In order to study a field variation, one must adopt a study reference system. For simplicity, this system will be linked to a coordinate system. It will then be possible, at an instant arbitrarily considered to be initial, to determine the set of position vectors of the material particles composing the domain. The reference configuration is thus defined. In order to describe the movement of the domain with respect to this reference system, it is appropriate to use the variation law over time of the position vectors.

A vector field as a function of time is then obtained. This will give the actual configuration. By convention, we will use upper case characters for the reference configuration and lower case characters for the actual configuration.

It is to be noted that one must be able to give the variation laws of the present position parameters according to the initial position parameters and time so as to get to know the domain variation. The initial position parameters and time are Lagrange variables.

13.

The velocity vector of a point is obtained by derivation with respect to the position vector time. If the basis used is a Cartesian basis, i.e. with vectors independent from space and time, the components of the velocity vector in this basis will be simply obtained by derivation with respect to the time of the position vector components. Since in Lagrange variables, the point initial position is independent from time, one can use either a straight derivative or a partial derivative to calculate the velocity vector components.

For the acceleration vector, the calculations are just as simple as soon as one knows that it is sufficient to obtain the derivative of the velocity vector with respect to time. It is also possible to obtain two derivatives of the position vector.

14.

It is also possible to use the motion vector to express the previous quantities. The motion of a point represents the difference between the position vector in the actual configuration and the position vector in the reference configuration.

Given that the position vector in the reference configuration is independent from time *t*, it is then simple to demonstrate that the velocity vector is obtained by a simple derivation of the motion vector. For the acceleration vector, it will be necessary to obtain two derivatives.

15.

In the Lagrange description, the observer is linked to the reference configuration and follows the domain motion. But it is also possible to imagine that the observer is linked to the domain, hence to the actual configuration. This change of viewpoint is allowed for the continuity assumptions involve a bijection between both configurations. We will then determine the reference position variables with respect to the present position variables and time.

The calculation of the velocity vector starts with the same relation as for the Lagrangian interpretation, however great attention must be paid to the fact that the reference coordinates are time-dependent, as a result of the bijection. It is therefore essential to use a partial derivative with respect to time of the position vector components in order to obtain the velocity vector components.

Similar care must be taken to determine the acceleration vector components.

16.

In our studies, we will often consider the variations of different functions over time. These functions may be simply scalar, like volumetric mass, or vectorial like the motion vector. But, as will be seen later on, they may also take more complex forms that we will call tensors. For further information on these notions, the student is invited to find documents in the library, bookshops or on the Internet. As a matter of fact, despite the fact that these tensor notions are very useful for this course, time is too short to develop tensor algebra and analysis here. We will simply give the elements necessary to an understanding of the course, as and when required.

We will therefore consider a generic tensor magnitude denoted A. In order to differentiate the Lagrangian and Eulerian approaches, we will use different symbols. For the Lagrangian representation, our tensor function will depend on the reference coordinates and time, whereas for the Eulerian representation, it will depend on the present coordinates and time.

Concerning the variations of our magnitude with respect to time, we can consider them assuming that they are to be expressed in a geometrical point in space. This will be referred to as local derivative.

This is simply obtained with a partial derivative with respect to time, i.e. by considering that Lagrange or Euler spatial variables are fixed. In this case, both interpretations, Lagrangian and Eulerian, give the same result.

17.

We can also take a look at the variations of our tensor magnitude by considering that it is directly linked to a material point of our domain. This will then be referred to as a particle derivative. It is the derivative of our function linked to a particle traced as it moves along. This time a straight, not partial, derivative will be used. In other words we will have to take into account the fact that space variables can be time-dependent.

In the Lagrangian interpretation, this problem does not exist for the space variables used are the coordinates of the material point studied at the reference instant. They are totally independent from time. The particle derivative is then identical to the local derivative.

On the contrary, in the Eulerian interpretation, the space variables used are the point coordinates in the actual configuration, i.e. at the moment considered. Therefore they are directly time-dependent, unless the point used is motionless with respect to our reference system, which is far from being the case generally speaking. In the calculation of our particle derivative, it is necessary to take account of this variation of the space coordinates with respect to time. As a result, a complementary term has to be added to the local derivative in order to obtain the particle derivative. If we take a close look at this term, it is to be noted that it involves the

space variations of our magnitude on the one hand, and the temporal variations of the position coordinates of the point studied on the other hand.

Therefore we obtain the gradient components of our magnitude and the velocity vector components; this allows us to write our particle derivative in the Eulerian approach in a reduced form.

18.

For application purposes, it is possible to say for instance that the magnitude studied is the point velocity vector. The particle derivative will give the acceleration vector of course.

By applying the previous relationship, we obtain the calculation formula of the acceleration vector i, the Eulerian approach.

It is also possible to give an index form permitting to calculate the components of this acceleration vector.

Finally imagine you are in a car on a Friday evening during the holiday period at the entrance of the Fourvière tunnel in Lyon. As you are familiar with the usual congestion occurring at this time and place, you took the precaution not to leave too early. However, as you approach the tunnel from the north, the traffic slows down. Your speed decreases and your car speedometer indicates a deceleration (negative acceleration). However the on-looker who has been observing the scene from the side of the motorway for a couple of hours will be disappointed. As the congestion is clearing, the speed of passing cars is gradually increasing. As a result of this (positive) acceleration phenomenon, he will soon have nothing interesting to see. The first (negative) acceleration is that of a particle whose movement you follow. In the Lagrange variable, it is the particle derivative. With regard to the second (positive) acceleration, the movement is observed from a fixed point in space and only the time-dependent speed variations are considered: it is the local derivative.

19.

The tensor concept may seem abstract and that is why we are trying to illustrate this type of notion.

We did it when linking a vector with origin, intensity, direction and sense to a first order tensor. For each tensor of order 1, we made a bijection with a diagram.

This made it easier to understand the notion of addition of these tensors. In the case of our second order tensors, we will give two graphic representations.

$$\vec{A}=A_i\vec{E}_i \quad \vec{B}=B_i\vec{E}_i \qquad \vec{C}=\vec{A}+\vec{B}$$

20.

A second order tensor can be given by its representative in the eigenvector basis, this representative being then a diagonal matrix.

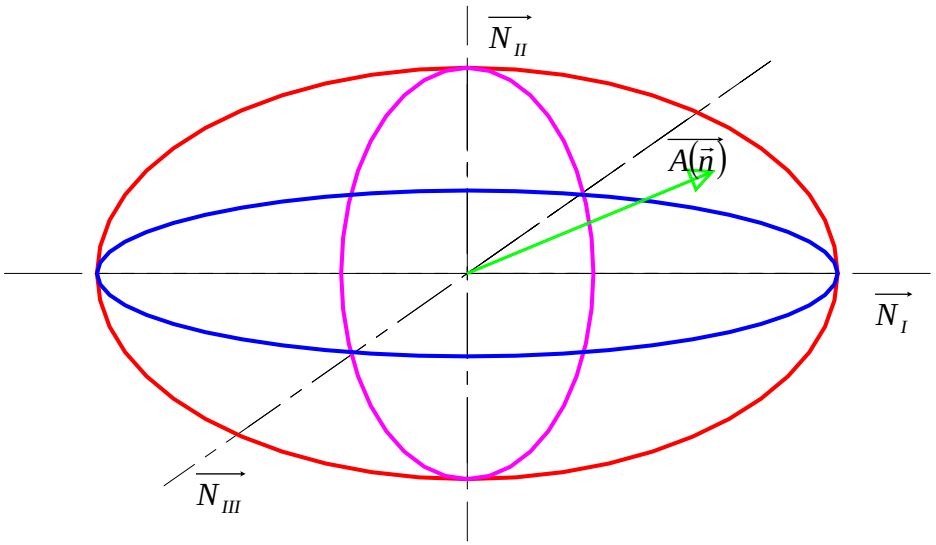
By means of this tensor application, to each space unit vector it is possible to link an image vector whose components can be calculated very simply in the eigenvector basis.

If one starts with a unit vector, one can see that there is a relationship between the image vector components. With this relationship, considering that the components of vector A represent the coordinates of a point in space, one can see that all the points possible are on an ellipsoid called Lamé ellipsoid.

This first three-dimensional graphic representation shows that the eigenvalues represent the extremal values of the tensorial state. Thus in the case of the deformation tensor, the greatest linear expansion and the smallest expansion at a point are given by two of the main deformations ε_I, ε_{II} and ε_{III}.

$$\mathbf{T}=\begin{pmatrix} T_I & 0 & 0 \\ 0 & T_{II} & 0 \\ 0 & 0 & T_{III} \end{pmatrix}_{(\overrightarrow{N_i})} \qquad \vec{n}=n_i\,\overrightarrow{N_i} \qquad \overrightarrow{A(\vec{n})}=\mathbf{T}\vec{n} \qquad \overrightarrow{A(\vec{n})}=A_i\,\overrightarrow{N_i} \qquad \begin{cases} A_1=T_I\,n_1 \\ A_2=T_{II}\,n_2 \\ A_3=T_{III}\,n_3 \end{cases}$$

$$n_1^{\,2}+n_2^{\,2}+n_3^{\,2}=1=\frac{A_1^{\,2}}{T_I^{\,2}}+\frac{A_2^{\,2}}{T_{II}^{\,2}}+\frac{A_3^{\,2}}{T_{III}^{\,2}}$$



21.

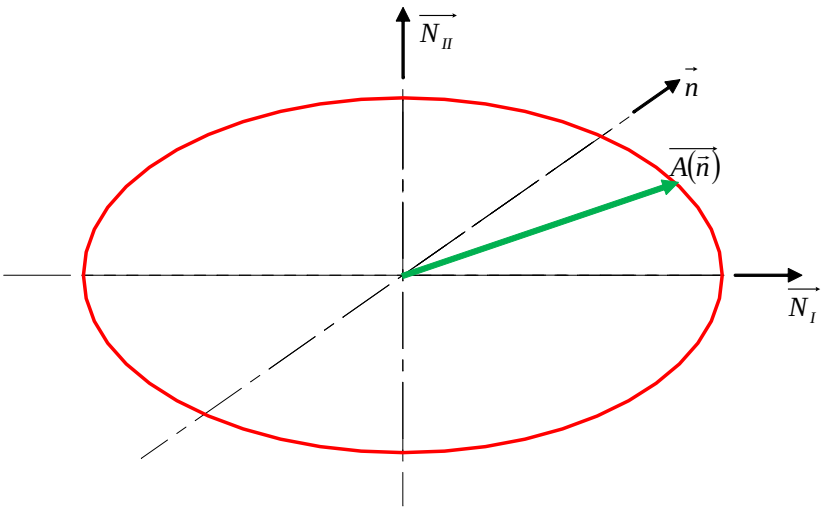
We will now see how to obtain graphically the construction of an ellipse resulting from the intersection of the ellipsoid with a main plane.

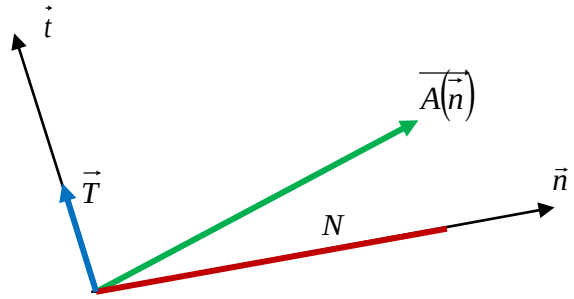
To do so, consider the plane formed by the study unit vector and its image vector by using the tensor application.

We denote by N the projection of the image vector onto the study vector.

The projection of the image vector onto the plane orthogonal to the study vector gives a vector called tangent vector $\vec{T}$, to which it is possible to link a unit vector $\vec{t}$.

From the study unit vector and its image vector, relations make it possible to calculate the normal vector and the tangent vector.





$$\begin{cases} \vec{N} = \left(\vec{n} \,.\, \overrightarrow{A(\vec{n})}\right)\vec{n} = N\,\vec{n} \\ \vec{T} = \vec{n} \wedge \overrightarrow{A(\vec{n})} \wedge \vec{n} = T\,\vec{t} \end{cases}$$

22.

Consider a unit vector belonging to a main plane and forming an angle alpha with a main direction $\vec{n} = \mathbf{cos}\,\alpha\,\vec{E}_I + \mathbf{sin}\,\alpha\,\vec{E}_{II}$

Its image can be calculated and defined by means of the tensor application.

23.

It is possible to give the components of this image vector in the eigenvector basis

$$\overrightarrow{A(\vec{n})} = \begin{pmatrix} T_I & 0 & 0 \\ 0 & T_{II} & 0 \\ 0 & 0 & T_{III} \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 = T_I \cos\alpha \\ A_2 = T_{II} \sin\alpha \\ 0 \end{pmatrix}_{\left(\overrightarrow{E_i}\right)}$$

24.

By gradually varying the angle between the main direction and the normal unit vector, one can see that the end of the image vector is shaped as an ellipse. It is to be noted that for this representation, the eigenvectors keep a fixed direction with respect to the observer.

25.

In order to enhance our knowledge of graphic representations, we will use an interactive module from the MECAGORA project. It makes use of the software VIRTOOLS. If it does not run on the computer, it will automatically install upon your initial login.

The mouse operating instructions are available at all times by clicking on the Information menu.

A classical screen displays several windows.

A graphic visualization window contains 3D objects which can be moved interactively using the mouse.

A control window in which it is possible to modify parameter values.

And a result window containing the numerical values of a number of calculated magnitudes.

26.

In Mécagora, module ia4 makes it possible to have a representation of Lamé ellipsoid according to the values given to the tensor application.

If it is represented by a diagonal matrix, the eigenvector basis is merged with the study basis.

27.

Lamé ellipsoid representation is three-dimensional, hence hard to use. To remedy this problem, we will use a plane representation. Using the basis change formulae it is possible to give the new components in the basis formed by the normal vector and the tangent vector:

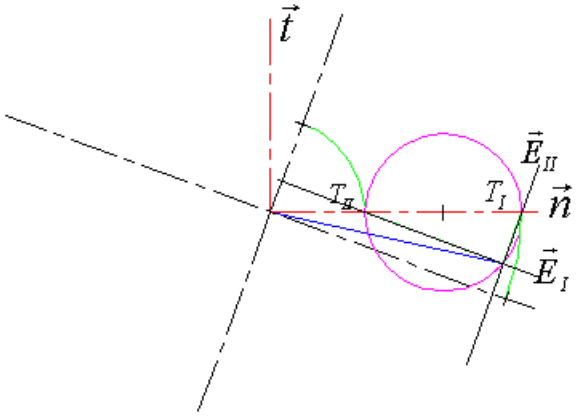
$$\overrightarrow{A(\vec{n})} = A_1 \vec{E}_I + A_2 \vec{E}_{II} = a_n \vec{n} + a_t \vec{t}$$

$$\begin{cases} a_n = T_I \cos^2\alpha + T_{II} \sin^2\alpha = \dfrac{T_I + T_{II}}{2} + \dfrac{T_I - T_{II}}{2}\cos(-2\alpha) \\ a_t = (T_{II} - T_I)\cos\alpha\sin\alpha = \dfrac{T_I - T_{II}}{2}\sin(-2\alpha) \end{cases}$$

And the trigonometry formulae make it possible to move to the double angle.

28.

These formulae show that when the angle varies, the end of the image vector is shaped as a circle called Mohr circle in the coordinate system linked to the normal unit vector.



29.

As for Lamé ellipse, this circle can be obtained by gradually varying the angle between the main direction and the normal unit vector. But unlike the construction of Lamé ellipse, the main directions are moving with respect to the observer.

30.

The plane containing the normal unit vector and its image vector will be called Mohr plane.

If the normal unit vector belongs to a main plane, the end of the image vector is located on a circle whose center is on the normal axis and whose intersection points with the normal axis have x-coordinates equal to the eigenvalues linked to the eigenvectors of the main plane T_I

As we generally have three different eigenvalues, we obtain a set of three circles called Mohr tri-circle.

31.

It is demonstrated that if the normal vector does not belong to any of the main planes, the end of the image vector is inside Mohr tri-circle.

The image vector projections onto the Mohr plane basis vectors give the normal vector and the tangent vector.

32.

In this figure, one can see that the greatest value of the normal vector is equal to the greatest eigenvalue;

the smallest value is that of the lowest eigenvalue.

Finally the greatest value of the tangent vector is equal to the radius of the largest circle, i.e. half the difference between the greatest eigenvalue and the smallest eigenvalue.